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INVARIANT SOLUTIONS OF RANK TWO OF THE EQUATIONS OF THE ROTATIONALLY-SYMMETRIC MOTIONS OF AN INHOMOGENEOUS LIQUID†

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An optimal system of first-order algebras of the system of equations for the rotationally-symmetric unsteady motion of an inhomogeneous liquid is constructed. New exact solutions of certain factor systems are found which describe motions with free boundaries or internal non-linear waves. © 1999 Elsevier Science Ltd. All rights reserved.

The problem of finding the invariant submodels of a certain system of differential equations of mechanics which admit a Lie algebra of operators reduces to constructing the optimal systems of subalgebras of this algebra of different orders. The method of finding the optimal systems of subalgebras has been described in [1, 2]. In the case of an infinite-dimensional Lie algebra, the optimal systems of subalgebras have been calculated [3, 4] in the case of the Navier–Stokes equations for the rotationally-symmetric motions of a homogeneous liquid and a three-dimensional Euler system. The optimal system of first-order subalgebras of the system of equations for the rotationally-symmetric motions of an inhomogeneous liquid is constructed below. Certain factor systems are integrated and a physical interpretation of the exact result obtained is given.

1. THE OPTIMAL SYSTEM OF SUBALGEBRAS Θ_1

The equations for the rotationally-symmetric motions of an inhomogeneous liquid in a cylindrical system of coordinates (r, θ, z)

$$\begin{aligned} &u_t + uu_r + wu_z - r^{-1}v^2 + \rho^{-1}p_r = 0 \\ &v_t + uv_r + wv_z + r^{-1}uv = 0 \\ &w_t + uw_r + ww_z + \rho^{-1}p_z = 0 \\ &\rho_t + u\rho_r + w\rho_z = 0, \quad u_r + r^{-1}u + w_z = 0 \end{aligned} \tag{1.1}$$

are considered. Here u is the radial component, v is the tangential component and w is the axial component of the velocity vector, ρ is the density of the liquid and p is the pressure (which are all functions of the variables (t, r, z)).

The basis of the Lie algebra which is admitted by system (1.1) consists of the operators‡

$$\begin{aligned} &X_1 = \partial_z, \; X_2 = t\partial_z + \partial_w, \; X_3 = \partial_t \\ &X_4 = 2r\partial_r + 2z\partial_z + t\partial_t + u\partial_u + v\partial_v + w\partial_w + 2p\partial_p \\ &X_5 = r\partial_r + z\partial_z + t\partial_t, \; X_6 = \rho\partial_\rho + p\partial_p \\ &X_7 = -r^{-2}v^{-1}\rho^{-1}\partial_v + r^{-2}\partial_p, \; X_8(\varphi) = \varphi(t)\partial_p \end{aligned} \tag{1.2}$$

The algebra is infinite-dimensional since the operator $X_8(\varphi)$ contains an arbitrary function $\varphi(t)$, we shall denote the Lie algebra with the basis of operators (1.2) by L.

One can use the equivalence transformation

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$$
\begin{gathered}
(t, r, z, u, v, w, \rho, p) \to (\bar{t}, r, \bar{z}, u, v, \bar{w}, \rho, p) \\
\bar{t} = t, \quad \bar{z} = z + gt^2/2, \quad \bar{w} = w + gt
\end{gathered}
\tag{1.3}
$$

to describe the unsteady motions of a liquid in a gravity field directed along the z axis, where g is the acceleration due to gravity. When this substitution is made, the structure of Eq. (1.1) is preserved and it is only on the right-hand side of the third equation that $-g$ appears. Every exact solution of Eqs (1.1) is translated into an exact solution of the equations of a heavy liquid by transform (1.3).

Sometimes, in constructing a factor system on an operator containing X_7, it is convenient to introduce a new function $h = (rv)^2$ which is the square of the angular momentum of a liquid particle around the z axis. In this case, we shall make use of another way of writing the first and second equations of system (1.1)

$$
u_t + uu_r + wu_z - r^{-3}h + \rho^{-1}p_r = 0, \quad h_t + uh_r + wh_z = 0 \tag{1.4}
$$

The operator X_7 acquires the simpler form

$$
X_7 = -2\rho^{-1}\partial_h + r^{-2}\partial_p
$$

In constructing the exact invariant solutions of system (1.1), we shall endeavour to ensure that these solutions (from the point of view of the admissible groups of transformations) are substantially different. Such solutions are obtained when solving factor systems constructed on operators from the optimal systems of algebras. The method of searching for the operators of the optimal systems of subalgebras has been described in detail in the papers by Ovsyannikov [1, 2].

In forming the optimal systems of subalgebras for the operators (1.2), the commutators of the operators were first calculated using the formulae [1]

$$
[X_i, X_j] = C_{ij}^k X_k = X_i(X_j) - X_j(X_i) \tag{1.5}
$$

Here C_{ij}^k are structural constants, $i, j, k = 1, \ldots, 8$ and the summation is over k. The structure of the algebra L of the operators (1.2) is investigated using the constants C_{ij}^k, and the associated group A of internal automorphisms of the algebra L is calculated.

To do this, an operator of the general form

$$
X = \sum_{i=1}^{7} x^i X_i + X_8(\varphi), \quad X \in L
$$

is considered, where $\mathbf{x} = (x^1, \ldots, x^7, \varphi)$ is a vector of the coordinates of the operator X in the basis (1.2). Automorphisms Aut_{X_i} of the algebra L are constructed on each operator $X_j \in L$ and the action of these automorphisms on X is found using the formula

$$
\mathrm{Aut}_{X_i}(a_i)(X) = X + \frac{a_i}{1!}[X, X_i] + \frac{a_i^2}{2!}[[X, X_i], X_i] + \ldots \tag{1.6}
$$

where $a_i (i = 1, \ldots, 7)$ are parameters. Formulae (1.6) determine the coordinates $\tilde{\mathbf{x}} = (\tilde{x}^1, \ldots, \tilde{x}^7, \varphi)$ of the transformed operator, which depend on the parameters a_i and the vector $\mathbf{x}$, generating the group A of automorphisms. The problem of finding the optimal system of subalgebras reduces to constructing the sets $\mathbf{x} = (x^1, \ldots, x^7, \varphi)$, which are such that none of the vectors can be translated into another vector by the autmorphisms of the algebra L [2].

The complete group of transformation of the coordinates of the vector

$$
\mathbf{x} \to \tilde{\mathbf{x}} = (\tilde{x}^1, \ldots, \tilde{x}^7, \tilde{\varphi}): \tag{1.7}
$$

$$
\begin{aligned}
&A_1 : \tilde{\mathbf{x}} = (x_1 - a_1(2x^4 + x^5), x^2, x^3, x^4, x^5, x^6, x^7, \varphi(t)) \\
&A_2 : \tilde{\mathbf{x}} = (x^1 + a_2x^3, x^2 - a_2x^4, x^3, x^4, x^5, x^6, x^7, \varphi(t)) \\
&A_3 : \tilde{\mathbf{x}} = (x^1 - a_3x^2, x^2, x^3 - a_3(x^4 + x^5), x^4, x^5, x^6, x^7, \varphi(t - a_3)) \\
&A_4 : \tilde{\mathbf{x}} = (e^{2a_4}x^1, e^{a_4}x^2, e^{a_4}x^3, x^4, x^5, x^6, e^{6a_4}x^7, e^{2a_4}\varphi(e^{-a_4}t)) \\
&A_5 : \tilde{\mathbf{x}} = (e^{a_5}x^1, x^2, e^{a_5}x^3, x^4, x^5, x^6, e^{2a_5}x^7, \varphi(e^{-a_5}t))
\end{aligned}
$$

$$A_6 : \tilde{\mathbf{x}} = (x^1,\ x^2,\ x^3,\ x^4,\ x^5,\ x^6,\ e^{a_6}x^7,\ e^{a_6}\varphi(t))$$
$$A_7 : \tilde{\mathbf{x}} = (x^1, x^2, x^3, x^4, x^5, x^6, x^7 - a_7(6x^4 + 2x^5 + x^6), \varphi(t))$$
$$A_8 : \tilde{\mathbf{x}} = (x^1, x^2, x^3, x^4, x^5, x^6, x^7, \varphi(t) - (2x^4 + x^6)\psi(t) + (x^3 + t(x^4 + x^5))\psi(t))$$

is determined in accordance with formulae (1.6). Here A_i is the transformation corresponding to Aut_{X_i} with the parameter $a_i (i = 1, \ldots, 7)$ and the transformation A_8 with the function $\psi(t)$ corresponds to $\mathrm{Aut}_{X_{(8)}(\varphi)}$. We conclude from the form of the transformations A_8 that, if $x^3 = x^4 + x^5 = x^6 + 2x^4 = 0$, then $\tilde{\varphi}(t) = \varphi(t)$ and, and if $x^3 \neq 0$ or $x^4 + x^5 \neq 0$ or $x^6 + 2x^4 \neq 0$, then a function $\psi(t)$ can always be found such that $\tilde{\varphi}(t) = 0$.

The operator X is determined, apart from an arbitrary multiplier; the transformation of general extension $B: \tilde{x} = \beta x$, $\beta = \text{const}$ is therefore admitted in the case of the vector $\mathbf{x}$.

On carrying out a structural analysis of the table of commutators, we note that $L = L_{1\text{–}7} \oplus L_\varphi$, where $L_{1\text{–}7} = \{X_1, \ldots, X_7\}$ is a finite-dimensional subalgebra and $L_\varphi = \{X_8(\varphi)\}$ is an infinite-dimensional subalgebra (the ideal of the algebra L). In turn, $L_{1\text{–}3} = \{X_1, X_2, X_3\}$ and $L_{6,7} = \{X_6, X_7\}$ are ideal in $L_{1\text{–}7}$. The following series of ideal is separated out and fixed

$$0 \subset \{L_\varphi\} \subset \{L_{6,7};\ L_\varphi\} \subset \{L_{1-3};\ L_{6,7};\ L_\varphi\} \subset L \tag{1.8}$$

According to (1.8), a decomposition of the algebra $L = L_{6,7,\varphi} \oplus L_{1\text{–}5}$ or $L = L_{1\text{–}3,6,7,\varphi} \oplus L_{4,5}$ is possible. This determines the sequence in which the subalgebras (or the coordinates of the vector $\mathbf{x}$) are treated and the action of internal automorphisms on them. We initially consider $L_{4,5}$ with the corresponding vector (x^4x^5). The versions (00), (x^40), $(0x^5)$, (x^4x^5), $x^4 \neq 0, x^5 \neq 0$ are possible in the case of (x^4x^5). Next, in $L_{1\text{–}5}$, we consider the vector $(x^1x^2x^3x^4x^5)$ depending on the vectors (x_4x_5) and the action of the group A of internal automorphisms (1.7). The result of this step is translated onto $L_{1\text{–}7}$ with a vector $(x^1, \ldots, x^7)$ taking account of the action of group A. Finally, we consider the algebra L with a vector $(x^1, \ldots, x^7, \varphi)$ and with the versions obtained for $(x^1, \ldots, x^7)$.

On taking account of further possible transformations from the group of automorphisms and the transformation of the general extension B of the vector $\mathbf{x}$, we obtain the optimal set of vectors $\mathbf{x}$. It is a set of substantially different vectors which cannot be translated into one another by the transformations of the group A of internal automorphisms.

The optimal system of subalgebras Θ_1 for Eqs (1.1), which corresponds to the set of coordinate vectors x, is

$$\begin{gathered}
\varepsilon X_1 + \delta X_7 + X_8(\varphi),\ X_2 + \varepsilon X_7 + X_8(\varphi),\ X_4 - X_5 - 2X_6 + X_8(\varphi) \\
\varepsilon_1 X_2 + X_3 + \varepsilon_2 X_7,\ \varepsilon X_2 + \nu X_3 + X_6,\ \nu X_2 + X_6 \\
\varepsilon X_1 + X_6,\ \varepsilon X_2 + X_5 + cX_6,\ \varepsilon X_2 + X_5 - 2X_6 + \nu X_7 \\
X_4 + bX_5 + cX_6,\ X_4 + bX_5 - 2(3 + b)X_6 + \nu X_7 \\
\nu X_3 + X_4 - X_5 + cX_6,\ \nu_1 X_3 + X_4 - X_5 - 4X_6 + \nu_2 X_7 \\
\nu X_1 + X_4 - 2X_5 + cX_6,\ \nu_1 X_1 + X_4 - 2X_5 - 2X_6 + \nu_2 X_7 \\
\delta = \{0;\ 1\};\quad \varepsilon, \varepsilon_1, \varepsilon_2 = \{-1;\ 0;\ 1\};\quad \nu, \nu_1, \nu_2 = \{-1;\ 1\}
\end{gathered} \tag{1.9}$$

where b and c are arbitrary constants and $\varphi(t)$ is an arbitrary smooth function.

Note that Eqs (1.1) admit of the following discrete transforms of their variables

$$(t, u, w) \to (-t, -u, -w),\ (\rho, p) \to (-\rho, -p)$$

$$(z, w) \to (-z, -w),\ (r, u) \to (-r, -u),\ v \to -v$$

although the second and fourth transforms do not have any physical meaning. The discrete transforms of the vector $\mathbf{x}$

$$E_1 : \tilde{\mathbf{x}} = (x^1, -x^2, -x^3, x^4, x^5, x^6, x^7, \varphi(-t))$$

$$E_2 : \tilde{\mathbf{x}} = (x^1, x^2, x^3, x^4, x^5, x^6, -x^7, -\varphi(t))$$

$$E_5 : \tilde{\mathbf{x}} = (-x^1, -x^2, x^3, x^4, x^5, x^6, x^7, \varphi(t))$$

where the transforms E_3 and E_4 are obtained as identity transforms, correspond to the given discrete transforms.

If we take account of the transforms E_1, E_2, E_5 in the optimal system of subalgebras (1.9), we obtain a set of operators of the form (1.9) in which the constants ε, ε_1, ε_2 have to be put equal to $\{0; 1\}$ and the constants ν, ν_1, ν_2 are equal to unity.

2. SOME EXACT SOLUTIONS

Here, we shall only present those factor systems which correspond to the subalgebras from (1.9) which admit of solutions in quadratures. As a rule, these are systems for which the continuity equation can be integrated. Since the pressure is determined apart from a term which is an arbitrary function of time, this function is not taken into account in the final representation for the pressure.

Example 1. We consider the first subalgebra from (1.9) when $\varepsilon \neq 0$, $\delta = 0$. The invariants of the operator under consideration determine the form of the solution

$$(u, v, w, \rho, p) = (U, V, W, R, \varepsilon z\varphi(t) + P)$$

and the functions U, V, W, R, P depend solely on (t, r). After substitution into (1.1), we obtain the equations

$$\begin{gathered} U_t + UU_r - r^{-1}V^2 + R^{-1}P_r = 0, \quad V_t + UV_r + r^{-1}UV = 0 \\ W_t + UW_r + \varepsilon\varphi(t)R^{-1} = 0, \quad R_t + UR_r = 0, \quad (rU)_r = 0 \end{gathered} \tag{2.1}$$

The last equation of (2.1) is integrated, and $U = C(t)/r$ with an arbitrary function $C(t)$.

We now introduce the Lagrangian coordinate ξ using the solution of the Cauchy problem

$$dr/dt = U, \quad r|_{t=0} = \xi \tag{2.2}$$

Then,

$$r = \left[\xi^2 + 2\int_0^t C(\tau)d\tau\right]^{1/2} \tag{2.3}$$

It is now seen that the solution of the four equations of system (2.1) can be represented as

$$\begin{gathered} V = \xi V_0(\xi)r^{-1}, \quad W = -\frac{\varepsilon}{R_0(\xi)}\int_0^t \varphi(\tau)d\tau + W_0(\xi) \\ R = R_0(\xi), \quad P = -\int R_0(\xi)r_\xi[r_{tt} - \xi^2 V_0^2(\xi)r^{-3}]d\xi \end{gathered} \tag{2.4}$$

with the arbitrary smooth functions $R_0(\xi)$, $W_0(\xi)$, $V_0(\xi)$.

Hence, the solution of the initial system (1.1) is

$$\begin{gathered} u = r^{-1}C(t), \quad v = V(\xi, t), \quad w = W(\xi, t) \\ p = \varepsilon z\varphi(t) + P(\xi, t), \quad \rho = R_0(\xi); \quad \xi = \left[r^2 - 2\int_0^t C(\tau)d\tau\right]^{1/2} \end{gathered} \tag{2.5}$$

and the functions V, W, P are determined by Eqs (2.4).

When $C(t) = 0$, solution (2.5) describes the unsteady vortex jet flow with a free boundary $r(t) = \xi_1 = \text{const}$ (or motion in a tube) along which a pressure $q(z, t) = z\varphi(t) + P(\xi_1, t)$ is applied. If $C(t) \neq 0$, we obtained a description of the motion of a cylindrical shell with free boundaries $r_1(t) = r(\xi_1, t)$, $r_2(t) = r(\xi_2, t)$, $(\xi_1 > \xi_2)$, and $r(\xi, t)$ is determined from (2.3). In both cases, the liquid density has an arbitrary distribution over the radius: $\rho = R_0(\xi)$.

Remark. It can be verified that the factor system (2.1) admits of the equivalence transform

$$W \to W + \varepsilon\psi(t)R^{-1}, \quad \varphi(t) \to \varphi(t) - \psi'(t)$$

and the remaining functions and variables are not transformed. On accounts of this, transforms, taking

$$\psi = -\int_0^t \varphi(t)dt$$

can be obtained in order that $\varphi(t) = 0$ and formulae (2.4) and (2.5) are simplified.

Example 2. If $\varepsilon \neq 0$, $\delta \neq 0$ ($\varepsilon^2 = 1$), the solution of Eqs (1.1), when account is taken of (1.4), can be sought in the form

$$(u, h, w, \rho, p) = (U, H - 2\varepsilon\delta z R^{-1}, \quad W, R, P + \varepsilon z(\delta r^{-2} + \varphi(t)))$$

where U, H, W, R, P are functions of the variables (t, r). Here, it has been taken into account that $\varepsilon^2 = 1$. The factor system is then transformed to the form

$$U_t + UU_r - r^{-3}H + R^{-1}P_r = 0, \quad H_t + UH_r - 2\varepsilon\delta WR^{-1} = 0$$
$$W_t + UW_r + \varepsilon R^{-1}(\delta r^{-2} + \varphi(t)) = 0, \quad R_t + UR_r = 0, \quad (rU)_r = 0$$

This system, apart from the transform

$$H \to H - 2\varepsilon\delta\mu(t)R^{-1}, \quad W \to W - \varepsilon\mu'(t)R^{-1}$$
$$P \to P + \varepsilon\delta\mu(t)r^{-2}, \quad \varphi \to \varphi + \mu''(t)$$

has the general solution

$$\begin{aligned} &U = \frac{C(t)}{r(\xi,t)}, \quad H = \frac{2W_0(\xi)}{R_0(\xi)}t - \frac{2\varepsilon\delta}{R_0^2(\xi)}\int_0^t\int_0^\sigma \frac{1}{r^2(\xi,\tau)}d\tau d\sigma + H_0(\xi) \\ &W = W_0(\xi) - \frac{\varepsilon\delta}{R_0(\xi)}\int_0^t \frac{1}{r^2(\xi,\tau)}d\tau, \quad R = R_0(\xi) \\ &P = \int R_0(\xi)r_\xi\left[\frac{H(\xi,t)}{r^3(\xi,t)} - r_{tt}(\xi,t)\right]d\xi \end{aligned} \tag{2.6}$$

where $C(t)$, $R_0(\xi)$, $W_0(\xi)$, $H_0(\xi)$ are arbitrary *functions* and the function $r(\xi, t)$ is defined by (2.3). In this case, the solution of system (1.1) has the form

$$\begin{aligned} &u = U(\xi,t), \quad v = \frac{1}{r(\xi,t)}\left[H(\xi,t) - \frac{2\varepsilon z}{R_0(\xi)}\right]^{1/2} \\ &w = W(\xi, t), \quad \rho = R(\xi, t) = R_0(\xi), \quad p = \varepsilon\delta z r^{-2}(\xi, t) + P(\xi, t) \end{aligned} \tag{2.7}$$

The Lagrangian coordinate ξ is determined by the last equality of (2.5).

If $C(t) = 0$ ($u = 0$), then formulae (2.7) describe motion in a tube which exists, generally speaking, for a finite time and the derivatives of the velocity component $\upsilon(r, t)$ are destroyed.

Example 3. We will now write the factor system for the operator $\langle X_2 + \varepsilon X_7 + X_8(\varphi)\rangle$. If $\varepsilon = 0$, then the invariants of the operator determine the form of the solution

$$(u, v, w, \rho, p) = (U, V, W + zt^{-1}, R, P + \varphi(t)zt^{-1})$$

where the functions U, V, W, R, P depend on (t, r). After substitution into (1.1), we obtain the equations

$$U_t + UU_r - r^{-1}V^2 + R^{-1}P_r = 0, \quad V_t + UV_r + r^{-1}UV = 0$$
$$W_t + UW_r + t^{-1}W + (tR)^{-1}\varphi(t) = 0$$
$$R_t + UR_r = 0, \quad (rU)_r + t^{-1}r = 0$$

This system can also be integrated. Actually, apart from the transform

$$W \to W + \psi R^{-1}, \quad \varphi \to \varphi - (t\psi)'$$

we obtain

$$U = -\frac{r(\xi,t)}{2t} + \frac{C(t)}{r(\xi,t)}, \quad V = \frac{\xi V_0(\xi)}{r(\xi,t)}, \quad W = \frac{1}{t}W_0(\xi)$$

$$R = R_0(\xi), \quad P = -\int R_0(\xi) r_\xi \left(r_{tt} - \frac{\xi^2 v_0^2(\xi)}{r^3(\xi,t)} \right) d\xi$$

The function $r(\xi, t)$ is defined by the equality

$$r = \left[\frac{1}{t} \left(\xi^2 + 2\int_1^t \tau C(\tau) d\tau \right) \right]^{1/2} \tag{2.8}$$

The physical quantities are given by the formulae

$$\begin{aligned} &u = U(\xi, t), \quad v = \xi v_0(\xi) r^{-1}(\xi, t), \quad w = W(\xi, t) + t^{-1} z \\ &p = P(\xi, t), \quad \rho = R_0(\xi) \end{aligned} \tag{2.9}$$

When $C(t) = 0$, solution (2.9) describes the unsteady vortex jet flow with a free boundary $r(t) = \xi_1/\sqrt{t}$ along which a pressure $q(z, t) = z\varphi(t) + P(\xi_1, t)$ is applied. If $C(t) \neq 0$, we obtain the motion of a cylindrical layer with free boundaries

$$r_1(t) = r(\xi_1, t), \quad r_2(t) = r(\xi_2, t) \quad (\xi_1 > \xi_2)$$

Example 4. If $\varepsilon \neq 0$ the solution of Eqs (1.1) and (1.4) can be sought in the form

$$(u, h, w, \rho, p) = (U, H - 2z\varepsilon(tR)^{-1}, \; W + zt^{-1}, \; R, \; zt^{-1}(\varepsilon r^{-2} + \varphi(t)) + P)$$

where U, H, W, R, P are functions of the variables (t, r). In this case, the factor system is

$$U_t + UU_r - r^{-3}H + R^{-1}P_r = 0, \quad H_t + UH_r - 2\varepsilon W(tR)^{-1} = 0$$

$$W_t + UW_r + t^{-1}W + (tR)^{-1}(\varepsilon r^{-2} + \varphi(t)) = 0$$

$$R_t + UR_r = 0, \quad (rU)_r + rt^{-1} = 0$$

Apart from the transform

$$H \to H + 2\varepsilon\mu(t)R^{-1}, \quad W \to W - t\mu'(t)R^{-1}$$

$$P \to P + \varepsilon\mu(t)r^{-2}, \quad \varphi \to \varphi + (t^2\mu'(t))'$$

the general solution has the form

$$U = -\frac{r(\xi,t)}{2t} + \frac{C(t)}{r(\xi,t)}, \quad H = H_0(\xi) - 2\left\{ \frac{1}{t}W_0(\xi) + \frac{\varepsilon}{R_0(\xi)} \int_1^t \frac{1}{\tau^2} \int_1^\tau \frac{d\sigma d\tau}{r^2(\xi,\sigma)} \right\}$$

$$W = \frac{1}{t}W_0(\xi) - \frac{\varepsilon}{R_0(\xi)} \int_1^t \frac{1}{r^2(\xi,\tau)} d\tau, \quad R = R_0(\xi)$$

$$P = \int R_0(\xi) r_\xi \left[\frac{H(\xi,t)}{r^3(\xi,t)} - r_{tt}(\xi,t) \right] d\xi$$

The function $r(\xi, t)$ is defined by (2.8). The components of the velocity vector, the density and the pressure are determined in this case using the formulae

$$u = U(\xi, t),\ v = \frac{1}{r(\xi, t)}\left[H(\xi, t) - \frac{2\varepsilon z}{R_0(\xi)}\right]^{1/2}$$
$$w = W(\xi, t) + \frac{z}{t},\ \rho = R_0(\xi),\ p = \frac{z\varepsilon}{r^2(\xi, t)t} + P(\xi, t) \tag{2.10}$$

This motion is similar to motion (2.7) and is destroyed after a finite time.

Example 5. In the case of the operator $\langle \nu X_2 + X_6 \rangle$ ($\nu^2 = 1$) from the optimal system of subalgebras (1.9), the invariant solution has the form

$$(u, v, w, \rho, p) = (U, V, W + zt^{-1}, \exp(\nu z t^{-1})R, \exp(\nu z t^{-1})P)$$

with the functions U, V, W, R, P of two variables (t, r). Equations (1.1) are written in the new variables as

$$U_t + UU_r - V^2 r^{-1} + R^{-1}P_r = 0,\ V_t + UV_r + r^{-1}UV = 0$$
$$W_t + UW_r + t^{-1}W + \nu(tR)^{-1}P = 0 \tag{2.11}$$
$$R_t + UR_r + \nu t^{-1}WR = 0,\ (rU)_r + rt^{-1} = 0$$

It is found that system (2.11) can be reduced to a single non-linear, non-classical third-order equation. Actually, as in Examples 3 and 4, we have

$$U = -\frac{r(\xi, t)}{2t} + \frac{C(t)}{r(\xi, t)},\ V = \frac{\xi V_0(\xi)}{r(\xi, t)}$$

The function $r(\xi, t)$ is given by (2.8). Furthermore, in the coordinates ξ, t

$$W = -\frac{\nu t R_t}{R},\ P = R\left(\frac{t^2 R_t}{R}\right)_t \tag{2.12}$$

Now, from the first equation of system (2.11), we obtain one equation in $R(\xi, t)$

$$\left[R\left(\frac{t^2 R_t}{R}\right)_t\right]_\xi + \left(r_{tt} - \frac{\xi^2 V_0^2}{r^3}\right) r_\xi R = 0$$

which, by means of the substitution $R = \exp Q$, $t = \tau^{-1}$, is reduced to the form

$$Q_{\tau\tau\xi} + Q_\xi Q_{\tau\tau} = \frac{\xi^3 V_0^2(\xi) + \xi C^2}{r^4(\xi, \tau)\tau} + \frac{\xi \tau C_\tau}{r^2(\xi, \tau)} - \frac{3}{4}\tau\xi \tag{2.13}$$

Equation (2.13) is somewhat simplified after the substitution $\xi \to \xi^2/2$. However, even for the case when $V_0 = 0$, $C = 0$, it is still not possible to construct a sufficiently wide class of exact solutions. Nevertheless, we shall point out one simple solution of Eq. (2.13) in the case when

$$Q = a\frac{\xi^2}{2} - \frac{\tau^3}{8a} + c_1\tau + c_2$$

where a, c_1, c_2 are constants. The interpretation of this solutions is the same as in Example 3.

Using the known function $Q(\xi, t)$, the physical quantities are recovered using the formulae

$$u = U(\xi, t),\ v = V(\xi, t),\ w = -\nu t Q_t + zt^{-1}$$
$$\rho = \exp(\nu z t^{-1} + Q),\ p = (t^2 Q_t)_t \exp(\nu z t^{-1} + Q)$$

Example 6. We now consider the subalgebra $\langle \varepsilon X_1 + X_6 \rangle$ when $\varepsilon \neq 0$. The invariant solution in this subalgebra must then be sought in the form

$$(u, v, w, \rho, p) = (U, V, W, R\exp(\varepsilon z),\ P\exp(\varepsilon z)) \tag{2.14}$$

where U, V, W, R and P depend solely on t, r (the case when $\varepsilon = -1$ corresponds to stable stratification). After substituting (2.14) into system (1.1), we obtain the factor system

$$\begin{aligned} &U_t + UU_r - r^{-1}V^2 + R^{-1}P_r = 0,\ V_t + UV_r + r^{-1}UV = 0 \\ &W_t + UW_r + \varepsilon R^{-1}P = 0,\ U_r + r^{-1}U = 0,\ R_t + UR_r + \varepsilon WR = 0 \end{aligned} \tag{2.15}$$

In turn, Eq. (2.15) can also be reduced to a single non-linear third-order equation which only differs from (2.13) in its right-hand side. We introduce the new independent variables and functions

$$\begin{aligned} &r = (\xi^2 + 2\int_0^t C(t)dt)^{1/2},\ U = \frac{C(t)}{r(\xi,t)},\ V = \frac{\xi V_0(\xi)}{r(\xi,t)} \\ &W = -\varepsilon Q_t,\ R = \exp Q,\ P = Q_{tt}\exp Q \end{aligned} \tag{2.16}$$

where $V_0(\xi)$, $C(t)$ are arbitrary functions. Then $Q(\xi, t)$ satisfies the equation

$$Q_{tt\xi} + Q_\xi Q_{tt} = B(\xi, t) \tag{2.17}$$

where

$$B(\xi, t) = \frac{\xi^3 V_0^2(\xi) + \xi C^2(t)}{r^4(\xi, t)} - \frac{\xi C_t(t)}{r^2(\xi, t)} \tag{2.18}$$

Hence, the solution of the form of (2.14) is completely defined if the function $Q(\xi, t)$ is known. The existence of a simpler right-hand side enables one to construct several exact solutions of Eq. (2.17).

We now assume that $C(t) = 0$ (there is no radial flow). In this case, solution (2.14) describes motion in a cylindrical tube, the function B in (2.18) depends solely on $r \equiv \xi$ and the substitution

$$\eta = \int B(\xi)d\xi$$

reduces (2.17) to the equation

$$Q_{tt\eta} + Q_\eta Q_{tt} = 1 \tag{2.19}$$

This equation has a solution in the form of a travelling wave

$$Q = F(\zeta),\ \zeta = \eta + \beta t,\ \beta = \text{const}$$

Here, F is the solution of the third-order equation

$$F''' + F'F'' = \beta^{-2}$$

which, by means of the substitution $F = \ln N^2$, can be reduced to a linear Airy equation

$$N'' - \left(\frac{\zeta}{2\beta^2} + c\right)N = 0$$

where c is a constant of integration. We have

$$N = \sqrt{\gamma}\left[C_1 I_{1/3}\left(\frac{2}{3}\gamma^{3/2}\right) + C_2 K_{1/3}\left(\frac{2}{3}\gamma^{3/2}\right)\right],\quad \gamma = (4\beta^4)^{1/3}\left(\frac{\zeta}{2\beta^2} + c\right) \tag{2.20}$$

where C_1, C_2 are constants and $I_{1/3}, K_{1/3}$ are modified Bessel functions. Note that ordinary Bessel functions have to be taken in formula (2.20) in the case of negative γ, and the solution will oscillate.

We now present the expression for the density

$$\rho = N^2 \exp(\varepsilon z)$$

and the lines of equal density are $z = -\varepsilon \ln N^2$.

In the case of other functions $C(t)$ which are non-zero, it is quite difficult to obtain solutions of Eq. (2.17) in explicit form. However, it is possible to choose $V_0(\xi)$ and $C(t)$ simultaneously when the right-hand side of Eq. (2.17) depends solely on ξ or is equal to zero. To do this, it is necessary to satisfy the equality

$$\xi^3 V_0^2(\xi)+\xi C^2(t)-\xi\left[\xi^2+2\int_0^t C(t)dt\right]C'(t)=B(\xi)\left[\xi^4+4\xi^2\int_0^t C(t)dt+4\left(\int_0^t C(t)dt\right)^2\right]$$

for any ξ, t. This equality is satisfied if

$$B(\xi)=B_0\xi,\ V_0^2=B_0\xi^2-\alpha^2\xi^{-2},\ C(t)=\alpha\sin(2\sqrt{B_0}t+\omega)$$

where $B_0>0$ and α, ω are constants. If $B_0<0$, then

$$V_0^2=B_0\xi^2+\alpha^2\xi^{-2},\ C(t)=\alpha\,\mathrm{sh}(2\sqrt{|B_0|}t+\omega)$$

Note that $B(\xi)=0$ only when $C(t)=\alpha t+\omega$, $V_0^2=\alpha-\omega^2\xi^{-2}$.

We now consider in somewhat greater detail the case when the right-hand side of Eq. (2.17) is equal to zero. When $B=0$, Eq. (2.17) reduces to the second-order ordinary differential equation

$$Q_{tt}-d(t)\exp(-Q)=0 \tag{2.21}$$

where $d(t)>0$ is a certain function. We now add the initial conditions

$$Q(\xi,0)=\ln R_0(\xi),\ Q_t(\xi,0)=-\varepsilon w_0(\xi) \tag{2.22}$$

to (2.21).

We assume that $d(t)=d_0>0$ is a constant; it is then possible to write the solution of the Cauchy problem (2.21), (2.22) in the form

$$Q=\ln\left\{\frac{1}{a}\mathrm{ch}^2\left[\sqrt{\frac{d_0a}{2}}t+\mathrm{arcch}(\sqrt{aR_0(\xi)})\right]\right\},$$

$$a=\frac{w_0^2(\xi)}{2d_0}+\frac{1}{R_0(\xi)}$$

For such a solution to exist, it is sufficient that the Froude number

$$\mathrm{Fr}=\frac{\min|w_0(\xi)|}{\sqrt{2d_0}}\geqslant 1$$

or $\min(1/R_0(\xi))\geqslant 1$.

We assume that $\varepsilon=-1$ (stable stratification) and in (2.21) put

$$d(t)=h\exp(ht^2/2),\ Q=D+ht^2/2;\quad h=\mathrm{const}>0$$

Then $D(\xi,t)$ is the solution of the Cauchy problem

$$D_{tt}=h(\exp(-D)-1),\ D(\xi,0)=\ln R_0(\xi),\ D_t(\xi,0)=w_0(\xi) \tag{2.23}$$

We now make the substitution

$$\sigma=\sqrt{h}t,\ D=Z+H(\xi);\quad H(\xi)=\frac{w_0^2(\xi)}{2h}+\frac{1}{R_0(\xi)}+\ln R_0(\xi)-1$$

The function Z then satisfies the equation

$$Z_\sigma^2/2=1-Z-\exp(-Z-H(\xi)) \tag{2.24}$$

It can be seen that, when $H>0$ (it is sufficient to require that the Froude number $\mathrm{Fr}\geqslant 1$), Eq. (2.24) has a solution which is periodic in σ, $Z_1 \leqslant Z \leqslant Z_2$, where $Z_1 < 0$, $Z_2 < 1$ are the roots of its right-hand side. Each liquid particle oscillates when its own period, since Z_1, Z_2 depends on ξ. Hence, in this case, solution (2.14) describes non-linear internal waves of various forms.

A further, simple solution of Eq. (2.17) is possible when $B = 0$. In fact, suppose that $Q_\xi = 0$ (the pressure vanishes when $Q_u = 0$). Then, in (2.16), it may be assumed that $Q(t)$ is an arbitrary function of time. Taking $Q = q(t)$, where $q(t)$ is a periodic function and $q_u > 0$, we obtain, in (2.14), a longitudinal velocity, density and pressure which are periodic in time.

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